

1. Consider the minimization problem: $\min |x| + |y|$ s.t. $x^2 + y^2 = 1$. By introducing binary decision variables to handle the absolute value terms, reformulate this problem as a mixed integer programming problem of the form.

2. Examine whether the following functions are convex or not.

$$x^2 + ax + b; x \in R$$

$$x^3; x \in R$$

$$x^4; x \in R$$

$$\log(x); x \in (0; 1]$$

3. Consider the quadratic function:

$$f(x) = 3x_1 + x_2 + 2x_3 + 4x_1^2 + 3x_2^2 + 2x_3^2 + (M-2)x_1x_2 + 2x_2x_3$$

For $M=0$ find the eigenvalues and eigenvectors and any stationary points. Are the stationary points local optima? global optima? Find the path of optimal solutions as M increases.

4. Prove that a Hessian matrix which is positive definite has positive eigenvalues.

5. Show that if B^k is positive definite, then $\cos \theta^k > 1/\kappa(B^k)$ where $\kappa(B^k)$ is the condition number of B^k , based on the 2-norm.

6. Derive a stepsize rule for the Armijo linesearch that minimizes the quadratic interpolant from the Armijo inequality.

Homework 1 Solution

1) Min $|x| + |y|$ s.t. $x^2 + y^2 = 1$

Let $x = x_+ - x_-$

and add $y = y_+ - y_-$
 and add $\beta_x \in \{0, 1\}$, $\beta_y \in \{0, 1\}$
 with

(a) $0 \leq x_+ \leq \beta_x$, $0 \leq x_- \leq 1 - \beta_x$

(b) $0 \leq y_+ \leq \beta_y$, $0 \leq y_- \leq 1 - \beta_y$

Reformulate to: Min $(x_+ + x_-) + (y_+ + y_-)$
 s.t. (a), (b)
 $(x_+ - x_-)^2 + (y_+ - y_-)^2 = 1$

- 2) $x^2 + ax + b$ - convex due to p.d. Hessian
 x^3 - nonconvex, Hessian not psd
 x^4 - convex, Hessian psd
 $\log(x)$ - nonconvex $\in (0, 1]$, Hessian not psd.

3) Min $[3 \ 1 \ 2]x + x^T \begin{bmatrix} 4 & \frac{M-2}{2} & 0 \\ \frac{M-2}{2} & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} x$

$$\det(H - \lambda I) = (4 - \lambda)(3 - \lambda)(2 - \lambda) - (4 - \lambda) - (M - 2)^2 \left(\frac{1}{2} - \frac{\lambda}{4}\right)$$

$$= \left(20 - \frac{(M-2)^2}{2}\right) + \frac{(M-2)^2 - 25}{4}\lambda + 9\lambda^2 - \lambda^3 = 0$$

for $M=2$, $\lambda = 1, 2.7, 3.0, 4.72$ (p.d.)

3.

<u>M</u>	<u>λ</u>
0	1.27, 3, 4.72
1	1.34, 3.36, 4.3
2	1.38, 3.6, 4.0
3	1.34, 3.36, 4.3
4	1.27, 3, 4.72
5	1.10, 2.70, 5.20
6	0.85, 2.48, 5.67

(symmetric about $M=2$)

positive definite for $M \in (-4.324, 8.324)$

Solutions

$$\begin{bmatrix} 4 & M-2/2 & 0 \\ M-2/2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} x = - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$Bx^* = -a$$

Solutions for x^* for various values of $M \in [0, 6]$ are given on next page.

4.

Hessian is symmetric ($H = H^T$)

and can be decomposed as:

$$H = V \Lambda V^T \text{ where } V^{-1} = V^T$$

Therefore if $p^T H p > 0$, then

$$0 < p^T H p = p^T V \Lambda V^T p = \bar{p}^T \Lambda p$$

$$= \sum \lambda_i \bar{p}_i^2 \text{ where } \bar{p} = V^T p$$

Now if $\lambda_i \leq 0$, then choosing $\bar{p} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$
 $= e_i$ leads to $0 < \lambda_i e_i^T e_i = \lambda_i \leq 0$
 which is a contradiction.

```

3 2 1
3 3 2
m = 5
: x a :=
1 -0.967742 3
2 0.580645 1
3 -1.29032 2
;
```

```

b :=
1 1 4
1 2 1.5
1 3 0
2 1 1.5
2 2 3
2 3 1
3 1 0
3 2 1
3 3 2
m = 6
: x a :=
1 -1.25 3
2 1 1 2
3 -1.5 2
;
```

```

b :=
1 1 4
1 2 2
1 3 0
2 1 2
2 2 3
2 3 1
3 1 0
3 2 1
3 3 2
;
```

```

1 1 4
1 2 0
1 3 0
2 1 0
2 2 3
2 3 1
3 1 0
3 2 1
3 3 2
;
```

```

m = 3
: x a :=
1 -0.769231 3
2 0.153846 1
3 -1.07692 2
;
```

```

b :=
1 1 4
1 2 0.5
1 3 0
2 1 0.5
2 2 3
2 3 1
3 1 0
3 2 1
3 3 2
;
```

```

m = 4
: x a :=
1 -0.833333 3
2 0.333333 1
3 -1.16667 2
;
```

```

b :=
1 1 4
1 2 1
1 3 0
2 1 1
2 2 3
2 3 1
3 1 0
;
```

```

m = 0
: x a :=
1 -0.833333 3
2 -0.333333 1
3 -0.833333 2
;
```

```

b :=
1 1 4
1 2 -1
1 3 0
2 1 -1
2 2 3
2 3 1
3 1 0
3 2 1
3 3 2
;
```

```

m = 1
: x a :=
1 -0.769231 3
2 -0.153846 1
3 -0.923077 2
;
```

```

b :=
1 1 4
1 2 -0.5
1 3 0
2 1 -0.5
2 2 3
2 3 1
3 1 0
3 2 1
3 3 2
;
```

```

m = 2
: x a :=
1 -0.75 3
2 2.77556e-17 1
3 -1 2
;
```

```

b :=
```

5. B^k is positive def.

$$\cos \theta^k = \frac{-\nabla f(x^k)^T p}{\|\nabla f(x^k)\| \|p\|}$$

- a few identities

$$p^T B^k p = \|B^{1/2} p\|^2, \quad \|B^{1/2}\|^2 = \|B^k\|$$

$$B^k p = -\nabla f(x^k)$$

$$-\nabla f(x^k)^T p = p^T B^k p = \|B^{1/2} p\|^2$$

- norm inequalities

$$p = (B^{-1/2}) B^{1/2} p$$

$$\|p\| \leq \|B^{-1/2}\| \|B^{1/2} p\| \Rightarrow \|B^{1/2} p\| \geq \frac{\|p\|}{\|B^{-1/2}\|}$$

$$\text{So } \cos \theta^k = \frac{p^T B^k p}{\|B^k p\| \|p\|} \geq \frac{\|B^{1/2} p\|^2}{\|B^k\| \|p\|^2}$$

$$\geq \frac{\|p\|^2}{\|B^k\| \|B^{-1/2}\|^2 \|p\|^2} = \frac{1}{\|B^k\| \|B^{-1/2}\|^2}$$

$$= 1/\kappa(B^k)$$

6. Quadratic interpolation

Armijo inequality not satisfied

$$f(x^k + \bar{\alpha} p) > f(x^k) + \nabla f(x^k)^T p (\gamma \bar{\alpha})$$

$$a \alpha^2 + b \alpha + c \quad \text{for } \alpha \in (0, \bar{\alpha}]$$

$$\alpha = 0, \quad c = f(x^k)$$

$$\alpha = 0, \quad b = \nabla f(x^k)^T p$$

$$\alpha = \bar{\alpha}, \quad a \bar{\alpha}^2 + b \bar{\alpha} + c = f(x^k + \bar{\alpha} p)$$

$$\alpha_2 = \frac{-b}{2a} = \frac{-\nabla f(x^k)^T p \bar{\alpha}}{2(f(x^k + \bar{\alpha} p) - f(x^k) - \bar{\alpha} \nabla f(x^k)^T p)}$$