1. Consider the penalty function given by $\phi(x;\rho) = f(x) + \rho \sum_{j=1}^{m} \varphi(h_j(x))$, where $\varphi(\xi)$ is a smooth function of ξ where $\varphi(\xi) = 0$ if $\xi = 0$ and $\varphi(\xi) > 0$ if $\xi \neq 0$, with a finite value of ρ, and compare the KKT conditions of min $f(x)$ $s.t.$ $h(x) = 0$ with a stationary point of the penalty function. Argue why these conditions cannot yield the same solution.

2. Given a nonzero Newton step of the optimality conditions, where the KKT matrix has the correct inertia, show when this step is a descent direction for the ℓ_1 merit function $\phi_1(x;\rho) = f(x) + \rho\|h(x)\|_1$.

3. Show that the tangential step $d_t = Z^k p_Z$ and normal step $d_n = Y^k p_Y$, respectively, can be found from

$$\begin{bmatrix} W^k & A^k \\ (A^k)^T & 0 \end{bmatrix} \begin{bmatrix} d_t \\ v \end{bmatrix} = - \begin{bmatrix} \nabla f(x^k) \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{W} & A^k \\ (A^k)^T & 0 \end{bmatrix} \begin{bmatrix} d_n \\ v \end{bmatrix} = - \begin{bmatrix} 0 \\ h(x^k) \end{bmatrix}$$

where both KKT matrices have the correct inertia and $(Z^k)^T \tilde{W} Y^k = 0$.

4. Select solvers from the SQP, interior point and reduced gradient categories, and apply these to

$$\begin{aligned} \min \quad & x_1 + x_2 \\ s.t. \quad & 1 + x_1 - (x_2)^2 + x_3 = 0 \\ & 1 - x_1 - (x_2)^2 + x_4 = 0 \\ & 0 \leq x_2 \leq 2,\ x_3, x_4 \geq 0 \end{aligned}$$

Use $x^0 = [0, 0.1]^T$ and $x^0 = [0, 0]^T$ as starting points.

5. Select solvers from the SQP, interior point and reduced gradient categories, and apply these to

$$\begin{aligned} \min \quad & x_1 \\ s.t. \quad & (x_1)^2 - x_2 - 1 = 0 \\ & x_1 - x_3 - 0.5 = 0 \\ & x_2, x_3 \geq 0 \end{aligned}$$

Use $x^0 = [-2, 3, 1]^T$ as the starting point.