06-720 **Advanced Process Systems Engineering** **Spring 2011**

Homework 2 **Due: 1/26/11**

1. Consider the convex problem:

$$\min\ x_1$$
$$\text{s.t. } x_2 \le 0;$$
$$x_1^2 - x_2 \le 0$$

Show that this problem does not satisfy LICQ and does not satisfy the KKT conditions at the optimum.

2. Quasi-Newton Methods

In the derivation of the Broyden update, the following convex equality constrained problem is solved:

$$\min \|J^+ - J\|_F^2$$
$$s.t. \quad J^+ s = y$$

Using the definition of the Frobenius norm from Section 2.2.1, apply the optimality conditions to the elements of J^+ and derive the Broyden update: $J^+ = J + \frac{(y - Js)s^T}{s^T s}$.

3. NLP Reformulation

A widely used trick is to convert (4.1) to an equality constrained problem by adding new variables z_j to each inequality constraint to form: $g_j(x) - (z_j)^2 = 0$. Compare the KKT conditions for the converted problem with (4.1). Discuss any differences between these conditions as well as the implications of using the converted form within an NLP solver.

4. Find the solution to: Min x_1 s.t. $x_2 \le x_1^3$, $-x_1^3 \le x_2$
and show that it does not satisfy KKT conditions. Explain why.

5. Consider the NLP problem:

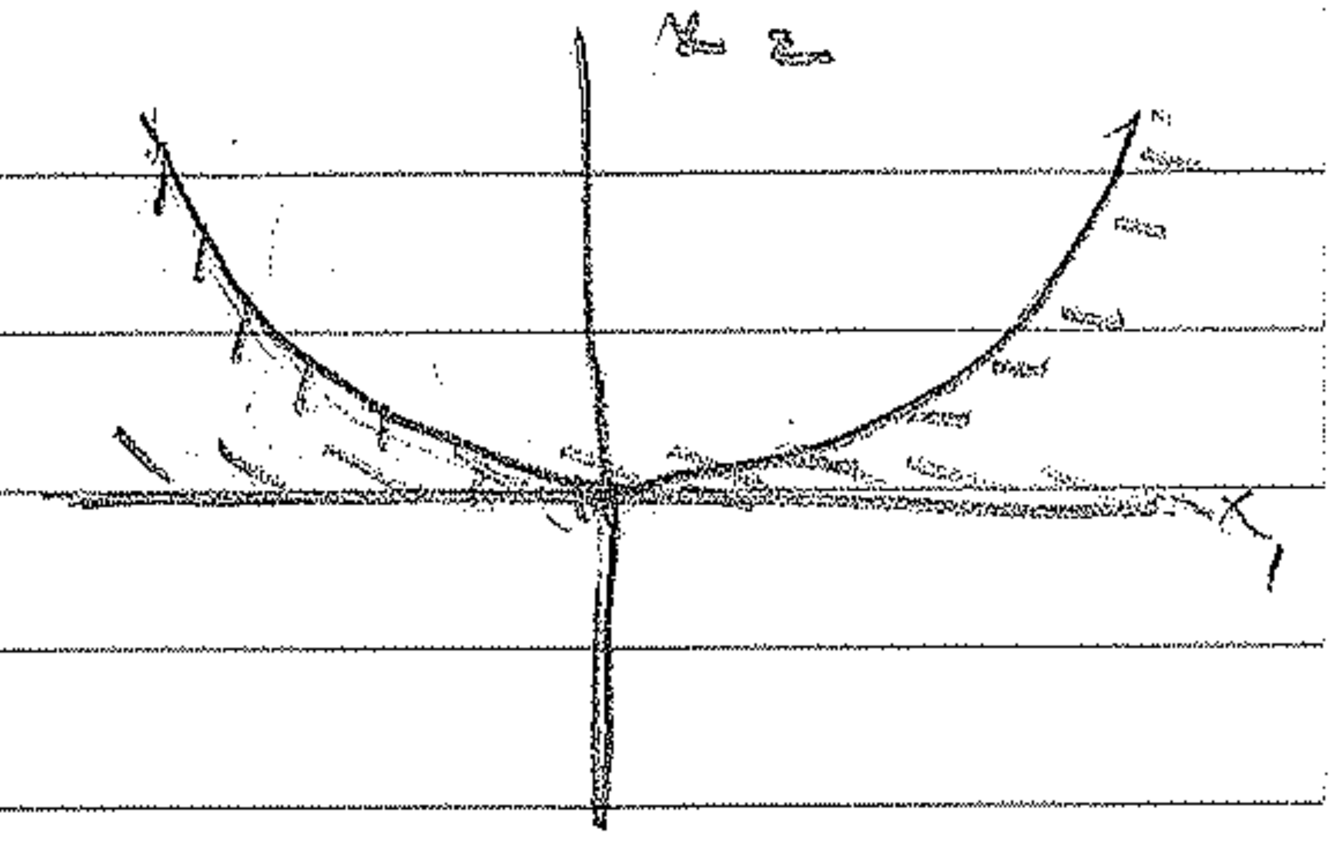
$$\text{Min } x_1 + (3x_2 - 1)^2$$
$$\text{s.t. } 2x_1 + x_2 - x_3 = 0$$
$$x_1, x_2, x_3 \ge 0$$

a) Write the first order KKT conditions and find the solution and multipliers for this problem.

b) At the solution of a) are the sufficient second order KKT conditions satisfied?

solution - HW 2

1) Min x_1
s.t. $x_2 \le 0$
$x_1^2 - x_2 \le 0$



KKT conditions
$\nabla f + \nabla g u = 0$
$g(x) \le 0, u^T g = 0, u \ge 0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 2x_1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

feasible region is only a single point (origin). At this point, $\nabla g(x^*)$ is singular and the KKT system has no solution.

2) $\min \| J^+ - J \|_F^2$
s.t. $J^+ s = y$

This problem can be rewritten as:

$$\min_{J^+} \sum_{ij} ((J^+)_{ij} - J_{ij})^2$$

$$\text{s.t. } \sum_j (J^+)_{ij} s_j = y_i$$

Optimality conditions are:

$$\cancel{L} = \sum_i \left\{ \sum_j (J^+_{ij} - J_{ij})^2 + \lambda_i \left(\sum_j J^+_{ij} s_j - y_i \right) \right.$$

$$\nabla \mathcal{L} = 2\left[(J^+_{ij} - J_{ij}) + \lambda_i s_j\right] = 0$$

or $(J - J^+) = \lambda s^T$

and from

$$J^+ s = (J - \lambda s^T)s = y$$

we know $\lambda = \dfrac{Js - y}{s^T s}$

leading to $J^+ = J + \dfrac{(y - Js)\, s^T}{s^T s}$

3. Reformulate to:

$$\text{Min } f(x) \text{ s.t. } g_j(x) - z_j^2 = 0,$$
$$h(x) = 0$$

KKT conditions are

$$\nabla f(x) + \nabla g(x) u + \nabla h(x) v = 0$$
$$2 z_j u_j = 0$$
$$g_j(x) = z_j^2$$
$$h(x) = 0$$

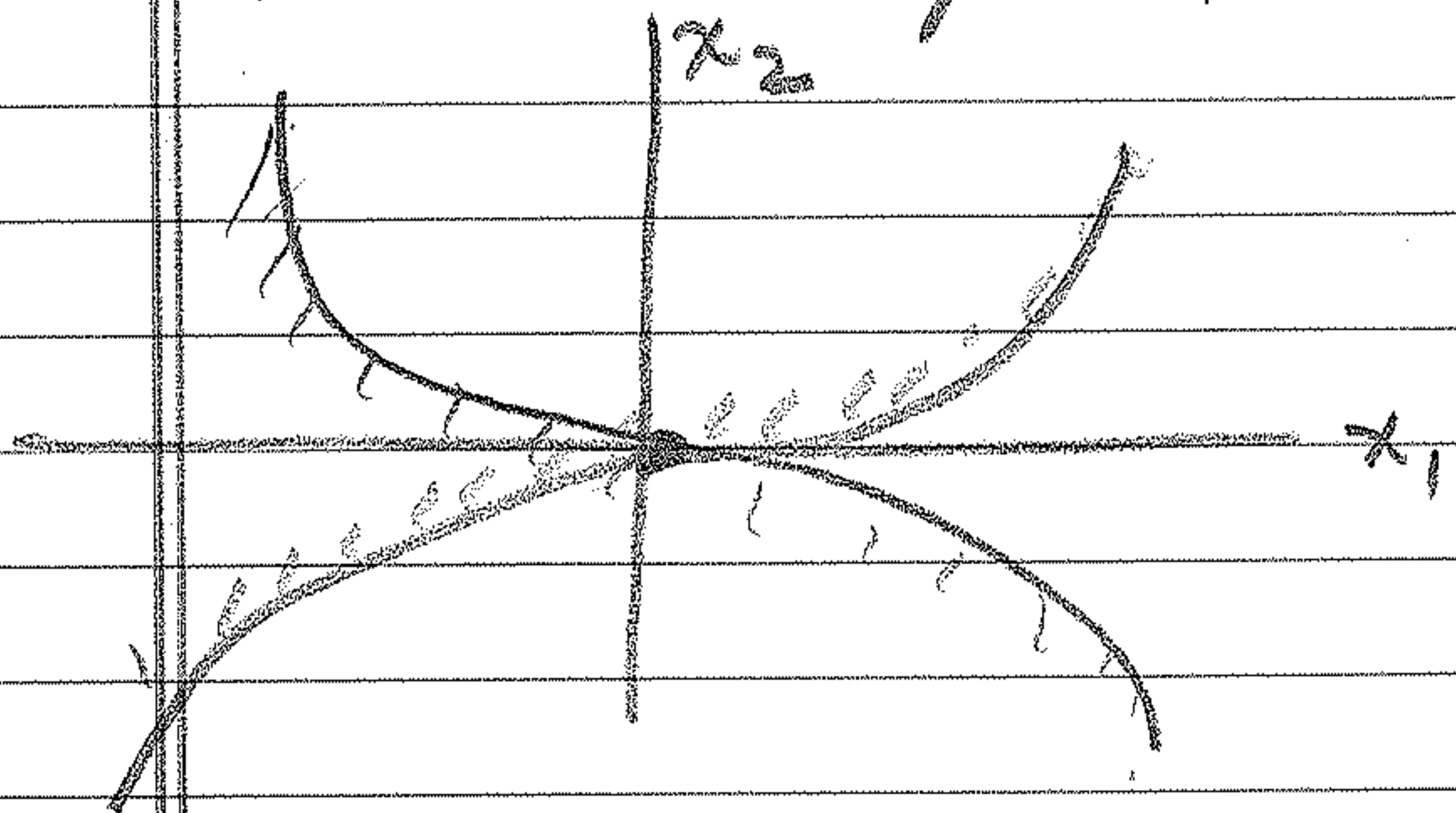
This approach is consistent w/ KKT conditions at KKT points. However, it also allows points where $g_j(x) = 0$ and $u_j < 0$. These points are suboptimal.

4. Min x_1 s.t. $x_2 \le x_1^3$, $-x_1^3 \le x_2$

KKT conditions: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_1^2 & -3x_1^2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$

LICQ not satisfied at $x = 0$.

This point is feasible and limit of x_1, hence optimal. But KKT conditions not satisfied.



5. $$\text{Min } x_1 + (3x_2 - 1)^2$$
$$\text{s.t. } 2x_1 + x_2 - x_3 = 0$$
$$x_1, x_2, x_3 \geq 0$$

a) KKT conditions

$$\nabla_x L = \begin{bmatrix} 1 \\ 18x_2 - 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} v - u = 0$$

$$0 \leq u \perp x \geq 0, \quad 2x_1 + x_2 - x_3 = 0$$

$$v = u_3$$
$$1 + 2v - u_1 = 0$$
$$18x_2 - 6 + v - u_2 = 0$$

Assume

$x_1 = 0$	$u_1 = 1$	KKT
$x_2 = 1/3$	$u_2 = 0$	conditions
$x_3 = 1/3$	$u_3 = 0$	satisfied
	$v = 0$	

b) Second order conditions, $u_1 > 0$

$$\nabla_{xx}\mathcal{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

allowable directions based on active set.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0$$

This corresponds to $p = Z\alpha$ where $\alpha \in \mathbb{R}$ and $Z = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$p^T \nabla_{xx} L p = \alpha^2 \left[Z^T \nabla_{xx} L Z \right]$$

$$= 18\alpha^2 > 0$$

Hence, sufficient second order conditions satisfied.