1. Consider the nonconvex, constrained NLP:

Min  \( (A + B) \)

\[
\begin{cases}
    x_1, y_1 \geq R_1 & x_1 \leq B - R_1, y_1 \leq A - R_1 \\
    x_2, y_2 \geq R_2 & x_2 \leq B - R_2, y_2 \leq A - R_2 \\
    x_3, y_3 \geq R_3 & x_3 \leq B - R_3, y_3 \leq A - R_3 \\
\end{cases}
\]

\[
\begin{cases}
    (x_1 - x_2)^2 + (y_1 - y_2)^2 \geq (R_1 + R_2)^2 \\
    \text{no overlaps} \quad (x_1 - x_3)^2 + (y_1 - y_3)^2 \geq (R_1 + R_3)^2 \\
    \quad (x_2 - x_3)^2 + (y_2 - y_3)^2 \geq (R_2 + R_3)^2 \\
\end{cases}
\]

\( x_1, x_2, x_3, y_1, y_2, y_3, A, B \geq 0 \)

Write the Kuhn-Tucker conditions for this problem.

a) Show that this problem is nonconvex
b) What can you say about the optimal active set of inequalities for this problem?

c) How does the system of Kuhn Tucker conditions lead to multiple NLP solutions?

2. Consider the NLP:

$$\text{Min } x_2$$

s.t.  
$$x_1 - x_2^2 + 1 \leq 0$$
$$-x_1 - x_2^2 + 1 \leq 0$$

a) Sketch the feasible region for this problem

b) Using GAMS, what happens if $$x_1 = x_2 = 0$$ is chosen as a starting point and SQP (SNOPT) or reduced gradient methods (CONOPT) are applied?

3. See the Brief Tutorial on GAMS, download GAMS from www.gams.com and refer to the CACHE file labeled alky.gms. Consider the alkylation process shown below from Bracken & McCormick (1968)

The alkylation is derived from simple mass balance relationships and regression equations determined from operating data. The first four relationships represent characteristics of the alkylation reactor and are given empirically.

The alkylate field yield, $$X_4$$, is a function of both the olefin feed, $$X_1$$, and the external isobutane to olefin ratio, $$X_8$$. The following relation is developed from a nonlinear
regression for temperature between 80 and 90 degrees F and acid strength between 85 and 93 weight percent:

\[ X_4 = X_1 \times (1.12 + .12167 \times X_8 - 0.0067 \times X_8^2) \]

The motor octane number of the alkylate, \( X_7 \), is a function of \( X_8 \) and the acid strength, \( X_6 \). The nonlinear regression under the same conditions as for \( X_4 \) yields:

\[ X_7 = 86.35 + 1.098 \times X_8 - 0.038 \times X_8^2 + 0.325 \times (X_6 - 89.) \]

The acid dilution factor, \( X_9 \), can be expressed as a linear function of the F-4 performance number, \( X_{10} \) and is given by:

\[ X_9 = 35.82 - 0.222 \times X_{10} \]

Also, \( X_{10} \) is expressed as a linear function of the motor octane number, \( X_7 \).

\[ X_{10} = 3 \times X_7 - 133 \]

The remaining three constraints represent exact definitions for the remaining variables. The external isobutane to olefin ratio is given by:

\[ X_8 = \frac{(X_2 + X_5)}{X_1} \]

To prevent potential zero divides it is rewritten as:

\[ X_8 \times X_1 = X_2 + X_5 \]

The isobutane feed, \( X_5 \), is determined by a volume balance on the system. Here olefins are related to alkylated product and there is a constant 22% volume shrinkage, thus giving \( X_4 = X_1 + X_5 - 0.22 \times X_4 \) or:

\[ X_5 = 1.22 \times X_4 - X_1 \]

Finally, the acid dilution strength \( X_6 \) is related to the acid addition rate \( X_3 \), the acid dilution factor \( X_9 \) and the alkylate yield \( X_4 \) by the equation, \( 1000 \times X_3 + X_4 \times X_6 \times X_9(98 - X_6) \). Again, we reformulate this equation to eliminate the division and obtain:

\[ X_6 \times (X_r \times X_9 + 1000 \times X_3) = 98000 \times X_3 \]

The objective function is a straightforward profit calculation based on the following data:

- Alkylate product value = $0.063/octane-barrel
- Olefin feed cost = $5.04/barrel
- Isobutane feed cost = $3.36/barrel
- Isobutane recycle cost = $0.035/barrel
- Acid addition cost = $10.00/barrel

This yields the objective function to be maximized is therefore the profit ($/day)

\[
OBJ = 0.063X4X7 - 5.04X1 - 0.035X2 - 10X3 - 3.36X5
\]

The following exercises are based on the description in Liebman et al (1984).

a) Set up this NLP problem and solve.

b) The regression equations presented in section 2 are based on operating data and are only approximations and it is assumed that equally accurate expressions actually lie in a band around these expressions. Therefore, in order to consider the effect of this band, Liebman et al (1984) suggested a relaxation of the regression variables. Replace the variables \(X4, X7, X9\) and \(X10\) with \(RX4, RX7, RX9\) and \(RX10\) in the regression equations (only) and impose the constraints:

\[
\begin{align*}
0.99X4 & \leq RX4 \leq 1.01X4 \\
0.99X7 & \leq RX7 \leq 1.01X7 \\
0.99X9 & \leq RX9 \leq 1.01X9 \\
0.9X10 & \leq RX10 \leq 1.11X10
\end{align*}
\]

to allow for the relaxation. Resolve with this formulation. How would you interpret these results?
alkey.gms

$ TITLE ALKYLATION PROBLEM FROM GINO USER'S MANUAL
$ OFFSYMXXREF
$ OFFSYMMLIST

OPTION LIMROW=0;
OPTION LIMCOL=0;
POSITIVE VARIABLES X1,X2,X3,X4,X5,X6,X7,X8,X9,X10;
VARIABLE OBJ;

EQUATIONS E1,E2,E3,E4,E5,E6,E7,E8;
E1..X4=E=X1*(1.12+.13167*X8-0.0067*X8**2);
E2..X7=E=86.35+1.098*X8-0.038*X8**2+0.325*(X6-89.);
E3..X9=E=35.82-0.222*X10;
E4..X10=E=3*X7-133;
E5..X8*X1=E=X2+X5;
E6..X5=E=1.22*X4-X1;
E7..X6*(X4*X9+1000*X3)=E=98000*X3;
E8.. OBJ =E= 0.063*X4*X7-5.04*X1-0.035*X2-10*X3-3.36*X5;

X1.LO = 0.;
X1.UP = 2000.;
X2.LO = 0.;
X2.UP = 16000.;
X3.LO = 0.;
X3.UP = 120.;
X4.LO = 0.;
X4.UP = 5000.;
X5.LO = 0.;
X5.UP = 2000.;
X6.LO = 85.;
X6.UP = 93.;
X7.LO = 90.;
X7.UP = 95.;
X8.LO = 3.;
X8.UP = 12.;
X9.LO = 1.2;
X9.UP = 4.;
X10.LO = 145.;
X10.UP = 162.;

X1.L =1745;
X2.L =12000;
X3.L =110;
X4.L =3048;
X5.L =1974;
X6.L =89.2;
X7.L =92.8;
X8.L =8;
X9.L =3.6;
X10.L =145;

MODEL ALKY/ALL/;
SOLVE ALKY USING NLP MAXIMIZING OBJ;
4. Power generation from Fuel Oil (CAChE Case Study by Prof. I. A. Karimi, NUS, file: fueloil.gms)

A two-boiler turbine-generator combination below is used to produce a power output of 30 MW with any combination of fuel oil and blast furnace gas (BFG). Only 10 units/h of BFG is available. Since the supply of BFG may not be sufficient for the required power generation, fuel oil must be purchased. It is desired to use the minimum amount of fuel oil in the two generators. Fuel requirements are expressed as a quadratic function of the power (MW) produced from a correlation:

\[ f = a_0 + a_1 x + a_2 x^2 \]

where \( x \) is power (MW) from each generator and \( f \) is fuel used (ton/h for fuel oil and units/f for BFG), with the constants for each generator given below.

![Diagram of power generation system]

<table>
<thead>
<tr>
<th>Generator</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>gen1(oil)</td>
<td>1.4609</td>
<td>.15186</td>
<td>.00145</td>
</tr>
<tr>
<td>gen1(gas)</td>
<td>1.5742</td>
<td>.16310</td>
<td>.001358</td>
</tr>
<tr>
<td>gen2(oil)</td>
<td>0.8008</td>
<td>.20310</td>
<td>.000916</td>
</tr>
<tr>
<td>gen2(gas)</td>
<td>0.7266</td>
<td>.22560</td>
<td>.000778</td>
</tr>
</tbody>
</table>

Assume that when a combination of fuel oil and BFG is used, the total power generated is summed. Power for first generator is from 18 to 30 MW while the second is 14 to 25 MW. Using the gams file fueloil.gms, go through the formulation of the problem. Identify the sets, variables, constraints and objective function. Also, not the special syntax of GAMS in referring to set indices.

a) What happens if gas supply increases from 10 to 15 units/h? Can you predict this from the multipliers?
b) Suppose that fuel oil supply is restricted to 10 ton/h and BFG is to be purchased. What is the minimum amount of BFG needed to supply the power requirement?
fueloil.gms

$TITLE Power Generation via Fuel Oil
$OFFUPPER
$OFFSYMXREF OFFSYMLIST

*OPTION SOLPRINT = OFF;
* Define index sets

SETS G Power Generators /gen1*gen2/
   F Fuels /oil, gas/
   K Constants in Fuel Consumption Equations /0*2/;

* Define and Input the Problem Data
TABLE A(G,F,K) Coefficients in the fuel consumption equations
       0       1       2
    gen1.oil  1.4609  .15186  .00145
    gen1.gas  1.5742  .16310  .001358
    gen2.oil  0.8008  .20310  .000916
    gen2.gas  0.7266  .22560  .000778;

PARAMETER PMAX(G) Maximum power outputs of generators /
   GEN1 35.0, GEN2 35.0/;

PARAMETER PMIN(G) Minimum power outputs of generators /
   GEN1 15.0, GEN2 15.0/;

SCALAR OilUP Maximum supply of BFG in units per h /10.0/;
   PREQ Total power output required in MW /50.0/;

* Define optimization variables
VARIABLES P(G) Total power output of generators in MW
   X(G,F) Power outputs of generators from specific fuels
   Z(F) Total Amounts of fuel purchased
   BFGPUR Total amount of fuel oil purchased;
POSITIVE VARIABLES P, X, Z;

* Define Objective Function and Constraints
EQUATIONS TPOWER Required power must be generated
   PWR(G) Power generated by individual generators
   BFGUSE Amount of oil purchased to be minimized
   FUELUSE(F) Fuel usage must not exceed purchase;

TPOWER.. SUM(G, P(G)) =G= PREQ;
PWR(G).. P(G) =E= SUM(F, X(G,F));
FUELUSE(F).. Z(F) =G= SUM((K,G), a(G,F,K)*X(G,F)**(ORD(K)-1));
BFGUSE. BFGPUR =E= Z("gas");

* Impose Bounds and Initialize Optimization Variables
  * Upper and lower bounds on P from the operating ranges
    P.UP(G) = PMAX(G);
    P.LO(G) = PMIN(G);
  * Upper bound on BFG consumption from GASSUP
    Z.UP("oil") = OILUP;
  * Specify initial values for power outputs
    P.L(G) = .5*(PMAX(G)+PMIN(G));
  * Define model and solve
    MODEL FUELOIL /all/;
    SOLVE FUELOIL USING NLP MINIMIZING BFGPUR;

DISPLAY X.L, P.L, Z.L, BFGPUR.L;