1. For $\alpha = 4$, and for $\alpha = 100$ solve the problem presented in class:

$$u'' - \alpha u = 4 t^2 - 2$$
$$u'(0) = 0, u(1) = 0$$

a) analytically
b) using finite differences with $h = 0.1$
c) using 2 pt. Collocation
d) single shooting

Legendre roots for collocation are given in the handout for $-1 \leq z \leq 1$ (and appended below). For this problem, we need to scale these roots between zero and one, so the collocation points can be found from: $t_i = 1/2(1 + z_i)$.

2. Apply 5 point global collocation to the problem below and solve:

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{\partial c}{\partial r}) = (c(r))^2$$

$$\frac{\partial c}{\partial r} = 0 \text{ at } r = 0$$
$$\frac{\partial c}{\partial r} = 100(c(1) - 1) \text{ at } r = 1$$

3. Solve problem 3 using finite differences with $h = 0.1$ and $h = 0.01$.

4. Solve problem 2 using COLSYS, COLNEW or COLDAE with a small number of collocation points (say, 3) and a small number of finite elements (say 5-10). To get these codes, go to http://www.netlib.org. The handout shows a demo of COLDAE with the FORTRAN subroutines to solve problem 1. This can be modified to solve problem 2. Subroutines for COLSYS or COLNEW are similar.
Legendre-Gauss quadrature is a numerical integration method also called "the" Gaussian quadrature or Legendre quadrature. A Gaussian quadrature over the interval \([-1, 1]\) with weighting function \(W(x) = 1\). The abscissas for quadrature order \(n\) are given by the roots of the Legendre polynomials \(P_n(x)\), which occur symmetrically about 0. The weights are

\[
\begin{align*}
\gamma_n &\quad = \frac{A_{n+1}}{A_n} \frac{\gamma_n}{P_n'(x_i)} P_{n+1}'(x_i) \\
&\quad = \frac{A_n}{A_{n-1}} \frac{\gamma_{n-1}}{P_{n-1}'(x_i) P_n'(x_i)},
\end{align*}
\]

where \(A_n\) is the coefficient of \(x^n\) in \(P_n(x)\). For Legendre polynomials,

\[
A_n = \frac{(2n)!}{2^n (n!)^2}
\]

(Hildebrand 1966, p. 323), so

\[
\frac{A_{n+1}}{A_n} = \frac{(2(n+1))!}{2^{n+1} ((n+1)!)^2} \frac{2^n (n!)^2}{(2n)!}
\]

\[
= \frac{n+1}{n+1}.
\]

Additionally,

\[
\gamma_n = \int_{-1}^{1} [P_n(x)]^2 \, dx
\]

\[
= \frac{2}{2n+1}
\]

(Hildebrand 1966, p. 324), so

\[
\begin{align*}
\gamma_n &\quad = \frac{2}{(n+1) P_{n+1}'(x_i) P_n'(x_i)} \\
&\quad = \frac{2}{n P_{n-1}'(x_i) P_n'(x_i)}
\end{align*}
\]

Using the recurrence relation

\[
(1-x^2) P'_n(x) = -nP_n(x) + nP_{n-1}(x)
\]

\[
= (n+1)xP_n(x) - (n+1)P_{n+1}(x)
\]

\[
\begin{align*}
\gamma_n &\quad = \frac{2}{(1-x^2) [P_n'(x_i)]^2} \\
&\quad = \frac{2}{(n+1)^2 [P_{n+1}'(x_i)]^2}
\end{align*}
\]

(Hildebrand 1966, p. 324).

The weights \(\gamma_i\) satisfy

\[
\sum_{i=1}^{n} \gamma_i = 2,
\]

which follows from the identity

\[
\sum_{i=1}^{n} \frac{1-x^2}{(n+1)^2 [P_{n+1}'(x_i)]^2} = 1.
\]

The error term is
\[ E = \frac{n^{2} \times (\sigma_{i}^{3})^{d}}{(2a+1)(2n)^{d}} \int_{0}^{2a+1} G(z) dz. \] 

(57)

Boyer (1987) gives a table of abscissas and weights up to \( n = 15 \), and Chandrasekhar (1960) up to \( n = 8 \) for \( n \) even.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_{i} )</th>
<th>( w_{i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \pm 0.57735 )</td>
<td>1.000000</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.6888890</td>
</tr>
<tr>
<td></td>
<td>( \pm 0.77497 )</td>
<td>0.6655689</td>
</tr>
<tr>
<td>4</td>
<td>( \pm 0.333983 )</td>
<td>0.6654158</td>
</tr>
<tr>
<td></td>
<td>( \pm 0.866138 )</td>
<td>0.6473055</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0.6638060</td>
</tr>
<tr>
<td></td>
<td>( \pm 0.518469 )</td>
<td>0.7810264</td>
</tr>
<tr>
<td></td>
<td>( \pm 0.90618 )</td>
<td>0.2569276</td>
</tr>
</tbody>
</table>

The exact abscissas are given in the table below.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_{i} )</th>
<th>( w_{i} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \pm \frac{1}{2} \sqrt{3} )</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>( \pm \frac{1}{2} \sqrt{5 - 2 \sqrt{3}} )</td>
<td>( \frac{1}{4} ) (18 + ( \sqrt{30} ))</td>
</tr>
<tr>
<td></td>
<td>( \pm \frac{1}{2} \sqrt{5 + 2 \sqrt{3}} )</td>
<td>( \frac{1}{4} ) (18 - ( \sqrt{30} ))</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{1}{3} \sqrt{245 - 14 \sqrt{70}} )</td>
<td>( \frac{1}{60} ) (322 + 13 ( \sqrt{70} ))</td>
</tr>
<tr>
<td></td>
<td>( \frac{1}{3} \sqrt{245 + 14 \sqrt{70}} )</td>
<td>( \frac{1}{60} ) (322 - 13 ( \sqrt{70} ))</td>
</tr>
</tbody>
</table>

The abscissas for order \( n \) quadrature are roots of the Legendre polynomial \( P_{n}(x) \), meaning they are algebraic numbers of degrees 1, 2, 2, 4, 6, 8, 10, 12, \ldots, which is equal to \( 2(n + 1) \) for \( n > 1 \) (Sloan's A003399).

Similarly, the weights for order \( n \) quadrature can be expressed as the roots of polynomials of degree 1, 3, 5, 7, 9, 11, \ldots, which is equal to \( 2(n + 1) \) for \( n > 1 \) (Sloan's A003399).

The triangle of polynomials whose roots determine the weights is

\[
\begin{align*}
216 x^2 & - 216 x + 49 \quad (21) \\
45,000 x^3 & - 32,200 x + 5,103 \quad (22) \\
202,500 x^4 & - 202,500 x^2 + 6,283,250 x - 58,564 \quad (23) \\
142,943,535,000 x^5 & - 113,071,253,400 x^3 + 27,510,743,799 x - 1,976,763,932 \quad (24) \\
1,707,698,764,800,000 x^6 & - 1,707,698,764,800,000 x^4 + 606,580,263,046,400 x^2 - 88,878,697,916,608 x + 4,573,849,350,625 \quad (25)
\end{align*}
\]

(Sloan's A112736)

REFERENCES:


